Lecture 24: Proof of Lovász Local Lemma

Recall: Lovász Local Lemma

Theorem

Let $(\mathbb{B}_1, \ldots, \mathbb{B}_n)$ *be the joint distribution of bad events. For each* $\mathbb{B}_i$*, where* $i \in \{1, \ldots, n\}$*, we have* $\mathbb{P}[\mathbb{B}_i] \leqslant p$ *and each event* $\mathbb{B}_i$ *depends on at most* d *other bad events. If* $\mathrm{e}p(d+1) \leqslant 1$*, then*

$$\mathbb{P}\left[\overline{\mathbb{B}_1}, \ldots, \overline{\mathbb{B}_n}\right] \geqslant \left(1 - \frac{1}{d+1}\right)^n > 0$$

The condition is also stated sometimes as $4pd \leqslant 1$ instead of $\mathrm{e}p(d+1) \leqslant 1$.

Proof of Lovász Local Lemma

Let us use an unproven claim to prove the Lovász Local Lemma

Claim

Let $S \subseteq 1, \ldots, n$ be an arbitrary subset. Then, we have

$$\mathbb{P}\left[\mathbb{B}_i \middle| \bigwedge_{k \in S} \overline{\mathbb{B}_k}\right] \leqslant \frac{1}{d+1}$$

Assuming this claim, it is easy to prove the Lovász Local Lemma.

$$\begin{aligned}\mathbb{P}\left[\bigwedge_{i=1}^{n} \overline{\mathbb{B}_i}\right] &= \prod_{i=1}^{n} \mathbb{P}\left[\overline{\mathbb{B}_i} \middle| \bigwedge_{k<i} \overline{\mathbb{B}_k}\right] \\ &\geqslant \prod_{i=1}^{n} \left(1 - \frac{1}{d+1}\right) = \left(1 - \frac{1}{d+1}\right)^n > 0\end{aligned}$$

Proof of the Claim

- We shall proceed by induction on $|S|$
- **Base Case.** If $|S| = 0$, then the claim holds, because

$$\mathbb{P}\left[\mathbb{B}_i \middle| \bigwedge_{k\in S} \overline{\mathbb{B}_k}\right] = \mathbb{P}\left[\mathbb{B}_i\right] \leqslant p \leqslant \frac{1}{\mathrm{e}(d+1)} \leqslant \frac{1}{d+1}$$

- **Inductive Hypothesis.** Assume that the claim holds for all $|S| < t$
- **Induction.** We shall now prove the claim for $|S| = t$. Suppose D_i be the set of all j such that the bad event $\mathbb{B}_i$ (possibly) depends on the bad event $\mathbb{B}_j$
- **Easy Case.** Suppose $S \cap D_i = \emptyset$. This is an easy case because

$$\mathbb{P}\left[\mathbb{B}_i \middle| \bigwedge_{k\in S} \overline{\mathbb{B}_k}\right] = \mathbb{P}\left[\mathbb{B}_i\right] \leqslant p \leqslant \frac{1}{\mathrm{e}(d+1)} \leqslant \frac{1}{d+1}$$

- **Remaining Case.** Suppose $S \cap D_i \neq \emptyset$.

$$
\begin{aligned}
\mathbb{P}\left[\mathbb{B}_i \,\middle|\, \bigwedge_{k \in S} \overline{\mathbb{B}_k}\right] &= \mathbb{P}\left[\mathbb{B}_i \,\middle|\, \bigwedge_{k \in S \cap D_i} \overline{\mathbb{B}_k}, \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}\right] \\
&= \frac{\mathbb{P}\left[\mathbb{B}_i, \bigwedge_{k \in S \cap D_i} \overline{\mathbb{B}_k} \,\middle|\, \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}\right]}{\mathbb{P}\left[\bigwedge_{k \in S \cap D_i} \overline{\mathbb{B}_k} \,\middle|\, \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}\right]} \\
&\leqslant \frac{\mathbb{P}\left[\mathbb{B}_i \,\middle|\, \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}\right]}{\mathbb{P}\left[\bigwedge_{k \in S \cap D_i} \overline{\mathbb{B}_k} \,\middle|\, \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}\right]} \\
&= \frac{\mathbb{P}\left[\mathbb{B}_i\right]}{\mathbb{P}\left[\bigwedge_{k \in S \cap D_i} \overline{\mathbb{B}_k} \,\middle|\, \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}\right]}
\end{aligned}
$$

- Our objective now is to lower-bound the denominator

Proof of the Claim III

- Suppose $S \cap D_i = \{i_1, \dots, i_z\}$
- Using the chain rule, we can write the denominator

$$\mathbb{P}\left[\bigwedge_{k \in S \cap D_i} \overline{\mathbb{B}_k} \middle| \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}\right]$$

as follows

$$\prod_{\ell=1}^{z} \mathbb{P}\left[\overline{\mathbb{B}_{i_\ell}} \middle| \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}, \bigwedge_{k' \in \{i_1, \dots, i_{\ell-1}\}} \overline{\mathbb{B}_{k'}}\right]$$

- Note that each probability term is condition on $< t$ bad events. So, we can apply the induction hypothesis. We get

$$\prod_{\ell=1}^{z} \mathbb{P}\left[\overline{\mathbb{B}_{i_\ell}} \,\middle|\, \bigwedge_{k \in S \setminus D_i} \overline{\mathbb{B}_k}, \bigwedge_{k' \in \{i_1, \dots, i_{\ell-1}\}} \overline{\mathbb{B}_{k'}}\right] \geqslant \prod_{\ell=1}^{z}\left(1 - \frac{1}{d+1}\right)$$

$$= \left(1 - \frac{1}{d+1}\right)^{z}$$

$$\geqslant \left(1 - \frac{1}{d+1}\right)^{d}$$

$$\geqslant \frac{1}{\mathrm{e}}$$

- Our goal of lower-bounding the denominator is complete. Let us return to our original expression

$$\mathbb{P}\left[\mathbb{B}_i \middle| \bigwedge_{k\in S} \overline{\mathbb{B}_k}\right] \leqslant \frac{\mathbb{P}\left[\mathbb{B}_i\right]}{\mathbb{P}\left[\bigwedge_{k\in S\cap D_i} \overline{\mathbb{B}_k} \middle| \bigwedge_{k\in S\setminus D_i} \overline{\mathbb{B}_k}\right]} \leqslant \mathrm{e}\mathbb{P}\left[\mathbb{B}_i\right] \leqslant \frac{1}{d+1}$$

- This completes the proof by induction
- We shall state and prove a more general result in the next lecture